Phase Noise Reduction in a MEMS Oscillator Using a Nonlinearly Enhanced Synchronization Domain

Oriel Shoshani, Daniel Heywood, Yushi Yang, Thomas W. Kenny, and Steven W. Shaw

Abstract—We investigate the phase dynamics of a closed-loop MEMS-based oscillator and demonstrate how one can exploit nonlinear behavior to improve oscillator phase noise characteristics by synchronizing the oscillator with a weak harmonic drive. Analytical predictions are based on an oscillator model that incorporates a resonator element with a weak cubic (i.e., Duffing type) nonlinearity, weak coupling to a clean external harmonic drive, and both thermal and frequency noise terms. The method of stochastic averaging is used to derive an expression for the rate of phase diffusion induced by the noises, and the results predict a remarkable phase noise reduction of three orders of magnitude when the oscillator is synchronized to the weak external field. The predictions are experimentally demonstrated using a closed-loop oscillator with a double-anchored double-ended-tuning-fork (DADETF) MEMS resonator with coupling to a small external sinusoidal signal from a signal generator, demonstrating a 30dB/Hz drop in phase noise. The results show how one can generate a clean high-power signal from a relative noisy oscillator by synchronizing it with a low-power external signal. The results also confirm recent studies showing that the parameter range of synchronization is significantly expanded when the resonator operates in its nonlinear regime.

Index Terms—Frequency stability, Nonlinear oscillators, Oscillator noise, Phase noise, Synchronization

I. INTRODUCTION

Due to their inherent compatibility with semiconductor technology and the ability to fulfill device miniaturization requirements, MEMS-based oscillators are an attractive replacement for quartz crystals in time-keeping applications [1]. They offer fast-responding, low power-consumption elements that are readily integrable with electronic circuits in fabrication [2]–[4]. However, as the size of MEMS devices are reduced, resonator frequency becomes more highly dependent upon fabrication variances and more susceptible to environmental effects, both of which result in degradation of frequency stability [5], [6]. Moreover, in order to produce large signal-to-noise ratio (SNR), these MEMS devices may need to operate nonlinearly [7], which leads to further degradation in the frequency precision due to amplitude-modulation (AM) to phase-modulation (PM) conversion [8]. In this paper, we describe an approach to reducing phase noise in a relatively noisy oscillator using coupling of a weak signal from a cleaner oscillator, essentially using synchronization to achieve clean signal amplification.

The synchronization of oscillators, a concept that goes back to Huygens [9], provides an approach for reducing phase noise or, equivalently, increasing frequency stability. This approach has been refined in other applications, e.g., synchronization of triode oscillators for radio communication systems [10], [11], synchronization of optical oscillators in laser systems [12], [13] and synchronization of MEMS oscillators for ultrasensitive resonant sensing [14], [15] and timekeeping [16], [17].

The practical significance of phase noise reduction due to synchronization has led to a large number of studies. Kurokawa [18] showed that the phase noise of an oscillator can be considerably improved by synchronization with an external signal, while the amplitude noise is slightly degraded. Chang et al. [19] showed the \( 1/N \) phase noise reduction in \( N \) mutually synchronized oscillators for a case where the amplitude noise and AM to PM conversion are negligible. Razavi [20] performed a theoretical and experimental investigation of injection locking, pulling, phase-locked oscillators, and phase noise reduction, in which he derived an identity for the requisite oscillator nonlinearity in order to obtain injection locking. Li et al. [21] presented discrete-time and frequency domain analyses of injection pulling and provided a detailed discussion on the benefits and deficiencies of each approach. Amro et al. [22] presented analysis for a pair of coupled oscillators with unequal noises. They showed that if the noisy oscillator is sufficiently noisy such that it is no longer synchronized with the clean oscillator, then it can actually improve the performance of the clean oscillator as the noisy oscillator becomes more noisy. Recent studies have experimentally demonstrated that nonlinearity in the frequency selective element (the resonator) of the oscillator can be exploited to manifest a regime in which the synchronization domain expands with increasing oscillation amplitude, thus providing a new strategy to enhance the synchronization of oscillators [23], [24].

Different aspects of frequency stability, including sensitivity to fluctuations of parameters, have been considered in numerous studies [25]–[27]. Rubiola and Giordano [28] and Sansa et al. [29] highlighted the importance of flicker noise frequency fluctuations as a limiting factor of the oscillator frequency short-term stability at relatively large time scales. The goal of this paper is to examine phase noise characteristics at small time scales of an externally-synchronized MEMS oscillator with a weakly nonlinear, lightly damped resonator element. While this synchronization is known to result in an
expanded synchronization domain, it also provides a dramatic reduction in the phase noise. To this end, we augment the investigations described in [23], [24] by including a stochastic analysis of the model and experimental tests with the following specific objectives: (i) to obtain a systematic and consistent derivation of a Langevin model for the oscillator phase; (ii) to use the model to provide theoretical predictions of the effective synchronization domain and the attendant phase noise reduction; and (iii) to demonstrate experimental validation of the theoretical predictions using a MEMS oscillator. The paper is organized as follows: We start with a description of the hardware setup in Section II, so that experimental comparisons can be made directly as analytical predictions are derived. Section III presents the derivation of a simplified Langevin equation (a noisy Adler equation) for the biased phase dynamics of a general noisy oscillator with a Duffing type resonator that is coupled to a weak external field. Section IV provides the specific predictions of interest from the model, namely the enhanced synchronization regime and the phase noise, along with a comparison of experimental results. Section V summarizes the main conclusions of the investigation and presents potential future research topics along similar lines.

II. EXPERIMENTAL SETUP AND DEVICE PROPERTIES

The device of interest is a closed-loop oscillator with a micro-mechanical frequency selective element, specifically, a double-anchored double-ended-tuning-fork (DADETF) resonator, as depicted in Fig. 1. The resonator consists of two beams 200 μm long and 6 μm thick connected on both ends to perforated masses, which are further anchored to a support base. Resonator parameters were estimated based on open-loop measurements [30], as follows: The natural frequency and quality factor are found to be: \( f_r = 1.188 \text{MHz} \) and \( Q = 7.013 \times 10^3 \), and the dimensionless Duffing parameter, which represents a stiffness term that is cubic in the displacement and arises from a combination of mechanical (hardening, arising from mid-line stretching of the beams [31]) and electrostatic (softening) is found to be: \( \alpha = 4.2 \times 10^{-3} \). Note that \( \alpha \) results in an amplitude-dependent frequency of free vibration [31], [32]. Oscillation was achieved by using a Zurich Instruments HF2LI lock-in amplifier to electrostatically drive the resonator at a constant phase shift. The output of the lock-in amplifier was maintained at a constant amplitude \( (R) \). A transimpedance amplifier and bandpass filter were used to condition the resonator output prior to measurement by the lock-in amplifier. The external signal was obtained from Agilent 33120A function generator and its amplitude \( (E) \) was kept at fixed ratio with respect to the oscillator amplitude \( (E/R = 8.33 \times 10^{-2}) \).

This setup provides self-sustained oscillation at a given amplitude that is then coupled to a weak external harmonic drive. The system is subject to unmodeled dynamics from a variety of sources that result in fluctuations in the response. The spectra of fluctuating quantities were measured using HP 89410A vector signal analyzer (VSA). In this work we do not consider the sources of these noises, but only their affect on the phase noise performance of the system, which is intimately related to frequency stability as the latter is simply time derivative of the former. We now turn to a model that describes these dynamics.

III. MODEL FORMULATION

We consider a model for a closed-loop oscillator with a high-Q weakly nonlinear (Duffing type) resonator element, thermal (additive) and frequency (multiplicative) noise sources, and an external harmonic drive. When expressed in terms of a dimensionless resonator displacement \( (x) \) the model is

\[
x + Q^{-1} x + x + \alpha x^3 = S(x, \dot{x}) + E \cos(\omega t) + \xi_1(t) + \xi_2(t) x \quad (1)
\]

where overdots denote derivatives with respect to the dimensionless time \( t = 2\pi f_r t_d \) \( (f_r \) is the resonator natural frequency and \( t_d \) is dimensional time), \( S \) represents the saturated closed-loop feedback input (gains, phase shifts, etc., for a specific model; see Agrawal et al. [16]), \( \alpha \) is the Duffing nonlinearity parameter, \( Q \) is the quality factor \( (Q \gg 1) \), \( E \) and \( \omega \) are the drive amplitude \( (E \ll 1) \) and drive angular frequency (which is close to the natural frequency, \( \omega = 1 - \Delta \omega \), \( \Delta \omega \ll 1 \)), and \( \xi_1, \xi_2 \) are wide-sense stationary, zero-mean, additive and multiplicative noises, respectively, which are also assumed to be weak \( (||\xi_1, \xi_2|| \ll 1) \). The resonator considered in this work is assumed to operate in an amplitude range for which a weakly nonlinear model holds, and the resonator in the experimental device has a hardening nonlinearity \( (\alpha > 0) \), the analysis presented is also applicable for a resonator with a softening nonlinearity \( (\alpha < 0) \).

In many cases the additive and multiplicative noises are used to model thermal and frequency fluctuations, respectively. Furthermore, these different noise sources can be associated with different correlation times and as a result be important at different time scales. For example, if the thermal and frequency fluctuations are white and flicker frequency noises, respectively, then the short term stability of the oscillator will be governed by thermal fluctuations (additive noise) at short times and by frequency fluctuations (multiplicative noise) at longer times. Here, however, we do not associate the additive and multiplicative noises with specific sources, but rather use...
them in a generic way to account for the combined effect of a variety of noise sources (fluctuation of temperature, frequency, feedback phase, quality factor, etc.) and consider only small time scales of the short term stability. Thus, we assume that the correlation times (τ_{ξ,2}) of ξ_{1,2} are sufficiently small when compared to the system relaxation time, τ_r = Q/(π f_r), i.e., τ_{ξ,2}/τ_r ≪ 1, or, equivalently, that the noises are wide-band compared to the generated signal. We then apply the method of stochastic averaging to obtain expressions that approximate the slow evolution of the oscillator amplitude a(t) and phase θ(t), where x(t) = a(t) cos(ωt + θ(t)). These are given by

\[
\dot{a} = s \frac{a}{2Q} - E \sin \theta + \frac{3S_{ξ_2}(2ω)}{16ω^2} a + \frac{S_{ξ_1}(ω)}{4ω^2} a^2 + \frac{1}{ω} \left( \frac{S_{ξ_1}(ω)}{2} + \frac{S_{ξ_2}(2ω)}{8} a^2 \right)^{1/2} \eta_1, \tag{2}
\]

\[
\dot{θ} = Δω + \frac{3α}{8ω} a^2 - \frac{E}{a} \cos θ - \frac{Ψ_{ξ_2}(2ω)}{8ω^2} + \frac{1}{ω} \left( \frac{S_{ξ_2}(2ω) + 2S_{ξ_2}(0)}{8} + \frac{S_{ξ_1}(ω)}{a^2} \right)^{1/2} \eta_2, \tag{3}
\]

where

\[
S_{ξ_2}(ω) + iΨ_{ξ_2}(ω) = 2\int_0^∞ \langle ξ_n(t)ξ_n(t + τ) \rangle \exp(-iωτ)dτ, \quad \langle η_{1,2}(t) \rangle = 0, \quad \langle η_{1,2}(t)η_{1,2}(t + τ) \rangle = δ_{nm}δ(τ), \tag{4}
\]

are the spectral values of ξ_{1,2} at angular frequency ω, according to the Wiener-Khinchin, theorem and η_{1,2} are simple delta-correlated Gaussian noises, according to the limit theorem of Stratonovich [33].

Note that in the absence of the external signal (E = 0) and noises (S_{ξ_n} = Ψ_{ξ_n} = 0), the closed-loop oscillator has a stable limit-cycle with an amplitude s = sQ ≡ R which is the ratio of the amplifier saturation level (s) to the system dissipation (Q^{-1}), which describes the condition for which the loop dynamics sustain the isolated oscillator. When the external signal and noises are small in comparison with the closed-loop gain (E, S_{ξ_1}/R, S_{ξ_2}/R ≪ s), we can rewrite Eq. (2) as

\[
\dot{a} = s \frac{a}{2Q} - \epsilon \frac{E}{a} \sin \theta - \frac{3sS_{ξ_2}(2ω)}{16ω^2} a - \frac{S_{ξ_1}(ω)}{4ω^2} a^2 - \frac{1}{ω} \left( \frac{S_{ξ_1}(ω)}{2} + \frac{S_{ξ_2}(2ω)}{8} a^2 \right)^{1/2} \eta_1, \tag{5}
\]

with \( E = E/ε, \ S_{ξ_1} = S_{ξ_1}/ε, \ E, \ S_{ξ_n} = O(1) \) and \( ε ≪ 1 \). Note that \( ε \) is an artificial book-keeping parameter introduced in order to emphasize that the external signal and noises are relatively weak compared to the deterministic part of the free-running oscillator. Thus, assuming that the perturbed amplitude is given by a = R + εr(t) + O(ε^2) and substituting this form into Eq. (5), we obtain the following governing equation for the amplitude perturbation away from R,

\[
\dot{r} = -\frac{r}{2Q} - E \sin \theta + \frac{3S_{ξ_2}(2ω)}{16ω^2} a - \frac{S_{ξ_1}(ω)}{4ω^2} a^2 + \frac{1}{ω} \left( \frac{S_{ξ_1}(ω)}{2} + \frac{S_{ξ_2}(2ω)}{8} a^2 \right)^{1/2} \eta_1. \tag{6}
\]

Note that in contrast to the phase dynamics, dissipation is explicitly present in the amplitude dynamics. Hence, perturbations of the amplitude decay relatively rapidly when compared on the time scale of the phase evolution. This point can also be seen from the free-running oscillator equations (Eqs. (2)-(3) with S_{ξ_n} = Ψ_{ξ_n} = E = 0), where the phase dynamics are associated with a zero eigenvalue and thus subject to slow evolution. Consequently, we assume that the amplitude is adiabatically following the phase and noises (\dot{r} ≈ 0), resulting in the perturbed amplitude

\[
a = R + \epsilon \left[ \frac{3QS_{ξ_2}(2ω)}{8ω^2} R + \frac{QS_{ξ_1}(ω)}{2ω^2} R - 2Q \frac{E}{ω} \sin \theta \right]
\]

\[
+ \frac{Q}{ω} \left( \frac{2S_{ξ_1}(ω)}{2ω^2} R^2 \right)^{1/2} \eta_1 + O(ε^2). \tag{7}
\]

We define the biased oscillator phase as \( φ = θ + θ_0, \ tan θ_0 = 2ω/(3αQR^2) \), which represents the phase difference between the external signal and the oscillator. Substitution of Eq. (7) into Eq. (3) and retaining terms up to O(ε^2), yields an approximated Langevin equation for φ given by

\[
\dot{φ} = Δω - A sin φ + ν(t), \tag{8}
\]

where

\[
Δω = Δω + \frac{3αR^2}{8ω} - \frac{Ψ_{ξ_2}(2ω)}{8ω^2} + \frac{3αQ}{32a^3}(4S_{ξ_1}(ω) + 3R^2S_{ξ_2}(2ω)), \tag{9}
\]

is the total frequency offset from all effects, and

\[
A = \frac{E}{R} \left( 1 + \left( \frac{3αQR^2}{2ω} \right)^2 \right)^{1/2}, \tag{10}
\]

is the mean phase modulation amplitude, and

\[
ν(t) = \frac{3αQR}{2ω^2} \left( \frac{S_{ξ_2}(ω)}{2ω^2} + \frac{R^2S_{ξ_2}(2ω)}{8} \right)^{1/2} \eta_1(t)
\]

\[
+ \frac{1}{ω} \left( \frac{S_{ξ_2}(ω) + 2S_{ξ_2}(0)}{8} + \frac{S_{ξ_1}(ω)}{R^2} \right)^{1/2} \eta_2(t). \tag{11}
\]

captures the net effects of the noises acting on the phase φ. Note that while Eq. (8) takes into account AM to PM conversion due to the Duffing nonlinearity, it solely describes the noisy phase dynamics and does not explicitly depend on the amplitude. A similar phase equation can be obtained using other methods, such as the perturbation projection vector (PPV) method [34], [35], orthogonal decomposition [26], and the method of isochrons and phase response curves [36]. However, in contrast to these alternative methods, our derivation yields the well-known noisy Adler equation (Eq. (8)) [37].
that appears in many other applications, including overdamped Josephson junctions [38], the theory of charge density waves [39], dithered-ring-laser gyroscopes [40], and vortex diffusion in layered superconductors [41].

IV. RESULTS

We now consider how this model is used to predict synchronization and phase noise performance in this class of systems and compare these predictions with results from the experimental device.

1) Nonlinear Synchronization Enhancement: We start the synchronization analysis with an investigation of the deterministic synchronized solution and the enhanced synchronization domain that stems from the resonator nonlinearity. The noise-free synchronization characteristics are determined from the deterministic part ($S_{\phi_0} = \Psi_{\phi_0} = 0$) of the Adler equation, Eq. (8) [42], which models the dynamics of an overdamped mass particle moving on a washboard potential.

The system fixed points $\phi^*$ which corresponds to a phase-locked/synchronized solutions

$$\sin \phi^* = \frac{\Delta\tilde{\omega}}{A}$$

where $\Delta\tilde{\omega} = \Delta \omega + \frac{3\alpha R^2}{8\omega}$ is the first order approximation of the frequency mismatch between the self-sustained oscillator angular-frequency $\omega_0 = 1 + \frac{3\alpha R^2}{8\omega}$ (i.e., the free vibration frequency of Duffing resonator) and the external signal frequency $\omega = 1 - \Delta \omega$.

For a frequency mismatch satisfying $-A < \Delta\tilde{\omega} < A$, the washboard potential is sufficiently flat such that the system has a pair of fixed points, one stable $\phi^*_s$ and one unstable $\phi^*_u$. Note that the condition $|\Delta\tilde{\omega}|_c = A$ defines the range of existence of these solutions, the stable of which represents synchronized response. This condition yields the following relation between the system parameters

$$|\Delta\tilde{\omega}|_c = \frac{E}{R} \left( 1 + \frac{3\alpha Q R^2}{2\omega} \right)^2$$

a result that is in agreement with those derived in [23], [24].

Following [24], two limiting cases of Eq. (12) are of interest, which have distinct scaling behavior with the amplitude $R$, as follows:

(i) A linear resonator element ($\alpha = 0$) which yields $2|\Delta\tilde{\omega}|_c = 2E/R = 2E/(sQ)$, so that, as expected, the frequency range of synchronization is proportional to the strength of the interaction with the external perturbation $E$ and the bandwidth of the resonant response ($Q^{-1}$).

(ii) A nonlinear resonator dominated by the Duffing term $(3\alpha Q R^2/(2\omega) \gg 1)$ which yields $2|\Delta\tilde{\omega}|_c \approx 3\alpha Q E R/\omega$.

Fig. 2 shows the analytical predictions and experimental measurements of the synchronization region width $2|\Delta\tilde{\omega}|_c$ for a fixed ratio of external perturbation amplitude to oscillator amplitude ($E/R = 8.33 \times 10^{-5}$), along with the analytical prediction calculated from Eq. (12), including the two limiting cases noted above. Note that all experimental measurements are in the range where the synchronization domains are significantly larger than their linear counterparts, that is, $3\alpha Q R^2/(2\omega) \gg 1$. Also note that in this range the biased phase $\phi$ and the oscillator phase $\theta$ coincide for all practical purposes, since $\phi = \theta + \theta_0$, $\tan \theta_0 = 2\omega/(3\alpha Q R^2)$.

That is, there there is no time delay between signals of the external perturbation and the synchronized oscillator.

2) Effects of Noise on Synchronization: The average phase velocity $\langle \dot{\phi} \rangle = \lim_{t \to \infty} \frac{\partial \langle \phi \rangle}{\partial t}$ and diffusion coefficient $D = \lim_{t \to \infty} \frac{1}{2t} \left( \langle \phi(t) - \langle \phi(t) \rangle \rangle^2 \right)$ of the noisy Adler equation Eq. (8) are the two quantities that characterize the dynamics, and these can be calculated analytically [33], [43]. The solution of the Adler equation can be interpreted as a random walk of an overdamped mass on the washboard potential. We consider the effective noise $\nu(t)$ to be delta-correlated $\langle \nu(t) \nu(t + \tau) \rangle = 2D_0 \delta(\tau)$ with intensity $D_0$ that is proportional to the spectral values $\langle S_{\phi_1,\tau}, \Psi_{\phi_1} \rangle$ of the (non-white) additive noise at the fundamental frequency and the (non-white) multiplicative noise at the second harmonic and at DC (see Eq. (10)). Fig. 3 shows the theoretical time derivative of the averaged phase $\langle \dot{\phi} \rangle$ versus the normalized detuning parameter for different levels of noise intensity $D_0$. Note that the mean phase rate $\langle \dot{\phi} \rangle$ captures the phase noise in the system and can be considered as the frequency deviation from the synchronized solution due to the presence of noise. The curves reveal that for sufficiently small values of the ratio of noise intensity to external-field coupling, specifically, for $0 < D_0/A < 0.005$, the noise has a negligible effect on the region of synchronization, resulting in phase locking in a noise-dependent range over the Shapiro step, that is, the segment of $\langle \dot{\phi} \rangle = 0$ where phase-locking occurs (this behavior is well known for Josephson junctions [44]); these cases are shown as the dotted, blue, and black curves in Fig. 3. In contrast, for relatively large noise intensities, e.g., $D_0/A \approx 1$, the noise completely eliminates the synchronization region and the time derivative of the averaged phase difference varies linearly with the frequency mismatch $(\Delta \omega)$, resulting in continual phase drift of a free running oscillator, as depicted
by the red line in Fig. 3. The transition between these limits is evident in Fig. 3, for example, as depicted by the green curve.

A. Phase Noise Analysis

The noisy phase dynamics can be determined from the model by linearization of Eq. (8) about the appropriate state and subtracting the mean steady-state value. For the synchronized response, which is achieved for \( D_\nu/A < 0.005 \), the model is linearized about the stable fixed point \( \langle \delta \phi \rangle = 0 \). This results in the following equation for the noisy phase deviation \( \langle \delta \phi \rangle = \phi - \langle \phi \rangle \)

\[
\delta \phi + (A^2 - \Delta \omega^2)^{1/2} \delta \phi = \nu(t).
\]

(13)

Note that, the phase deviation, \( \delta \phi \), is an Ornstein-Uhlenbeck process. Thus, the variance of the phase \( \langle \delta \phi^2 \rangle = \langle \phi^2 \rangle - \langle \phi \rangle^2 \) can be readily calculated from Eq. (13) and is given by

\[
\langle \delta \phi^2 \rangle = \frac{D_\nu}{\sqrt{A^2 - \Delta \omega^2}} \left[ 1 - \exp \left( -2(A^2 - \Delta \omega^2)^{1/2} t \right) \right].
\]

(14)

Thus, as the offset frequency \( \Delta \omega / A \) increases and the synchronized solution moves from the center \( \Delta \omega / A \ll 1 \) of the Shapiro step towards the edges \( \langle \Delta \omega / A \approx 1 \rangle \), both the variance of the locked solution \( \langle \delta \phi^2 \rangle_{\text{ss}} = \frac{D_\nu}{\sqrt{A^2 - \Delta \omega^2}} \) and the time it takes to achieve this value \( t_{\text{ss}} \propto (A^2 - \Delta \omega^2)^{-1/2} \) increase up to the boundary points of the synchronization domain, \( \Delta \omega / A = \pm 1 \), at which points Eq. (14) recovers the free-running solution behavior, \( \langle \delta \phi^2 \rangle = 2D_\nu t \). By taking the Fourier transform of Eq. (13) we obtain the following phase-noise spectrum

\[
\mathcal{L}_\phi(f) = 10 \log_{10} \left[ \frac{2D_\nu}{4\pi^2 f^2 + A^2 - \Delta \omega^2} \right].
\]

(15)

which again, at the boundary points of the synchronization domain, \( \Delta \omega / A = \pm 1 \), recovers the free-running solution spectrum

\[
\mathcal{L}_\phi(f) = 10 \log_{10} \left[ \frac{D_\nu}{2\pi^2 f^2} \right].
\]

(16)

Both \( \langle \delta \phi^2 \rangle \) and \( \mathcal{L}_\phi(f) \), which quantify the phase-noise in the time and frequency domains, respectively, show that the largest reduction in phase noise will occur at the center of the Shapiro step, \( \Delta \omega / A \ll 1 \). This result agrees with the intuition that the "overdamped mass" will remain in the well in spite of the random "kicks," so long as the washboard potential slope \( \langle \Delta \omega \rangle \) is not large compared to \( A \).

Fig. 4 shows a comparison between the theoretical predictions from Eqs. (15)-(16) and experimental measurements of the phase noise. The phase noise is computed for the given oscillator and external perturbation amplitudes of \( R_d = 300 \) mV, \( E_d = 25 \) mV and three different offset frequencies, \( f = 0.50, 1.0, 2.0 \) Hz. The effective noise intensity \( D_\nu \) is estimated from Eqs. (15)-(16) in order to achieve optimal agreement with the measurements (by means of a nonlinear least squares fit [45]) and yields the dimensional value \( D_{\nu,\text{phys}} \equiv 2\pi f D_\nu = 1.98 \times 10^{-6} \) Hz. This is the only parameter obtained from fitting experimental results to the model.

Note that the ratio of noise intensity to external field coupling yields \( D_\nu / A = 8.22 \times 10^{-6} \), which is considerably smaller than the theoretical upper bound of the synchronized solution \( (D_\nu / A)_{\text{max}} = 5 \times 10^{-3} \), thus confirming the validity of the theoretical model. Also, note that there is a considerable drop in the phase noise when the oscillator is synchronized, e.g., for \( f = 0.5 \) Hz there is a drop of 32 dB in the phase noise when the oscillator is synchronized, (e.g., \( f = 0.5 \) Hz) to \( (2\pi f, \Delta \omega) < 24.1 \) Hz) to \( (2\pi f, \Delta \omega) < 24.1 \) Hz, that is, the phase noise is reduced by three orders of magnitude in synchronized operation. The slight asymmetry in the experimental measurement may be due to the noise induced frequency drift \( (\Delta \omega - \Delta \omega - 2\omega R_d^2 / \kappa) = \frac{2\omega G \left( 4S_L (\omega) + 3R^2 S_L (\omega) \right) - \frac{\kappa}{\kappa}}{\kappa - 2\omega} \), which is not accounted for in our model prediction.

Fig. 5 shows the Allan deviation of the oscillator in both free-running and synchronized modes, along with that of the external harmonic signal. As it can be seen, the oscillator frequency stability is improved significantly over both short and long averaging times. The stability on short averaging times is governed by noises with small correlation time compared to the relaxation time of the oscillator, and this noise reduction is explained by our model. Of course, the noise reduction is limited by the quality of the external signal, which in the ideal
Figure 5. Allan deviation of the free-running oscillator (red), the synchronized oscillator with zero frequency mismatch (black), and the external harmonic signal (blue).

case serves as the noise floor at zero frequency mismatch.

V. CLOSING REMARKS

We derived and investigated a model for the phase noise of a MEMS based closed-loop oscillator that is synchronized by an external harmonic field. The case of weak external field and small intensity noise allowed us to reduce the problem to a single stochastic differential equation governing the phase dynamics of the synchronized oscillator. The corresponding Langevin equation for the oscillator phase is shown to be capable of predicting the remarkable phase noise reduction in the synchronized regime, along with the ability to enhance the synchronization region, by choosing a Duffing resonator as the frequency selective element, which allows for operation at relatively large oscillator amplitudes. The analytical predictions are validated by experimental measurements and provide useful insights into methods for reducing phase noise by exploiting system nonlinearities.

The most important feature of these results is that the system acts like a “clean” amplifier, specifically, one can use a low-power signal with good frequency stability to drive a noisy MEMS oscillator to produce a high-power signal that is significantly cleaner than that of the MEMS oscillator; in our device we obtain a twofold amplification of the injected signal.

In future work the analysis will be generalized to include the transition zones near the bifurcation points where the oscillator switches from non-synchronized to synchronized behavior. These regions are associated with phase slips which increase the phase noise considerably (see Fig. 4) and limit oscillator performance, and are of interest in other applications, such as quantum computing, where the readout of the qubits is done via a Josephson junction bifurcation amplifier that has similar characteristic transition dynamics.

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